Today's Lecture

- DAG structure
- Dynamic programming to find highest weight paths in WDAGs

Structure of DAGs

- Define the *depth* of a node v in V as:
  - the length of the longest path ending at v;

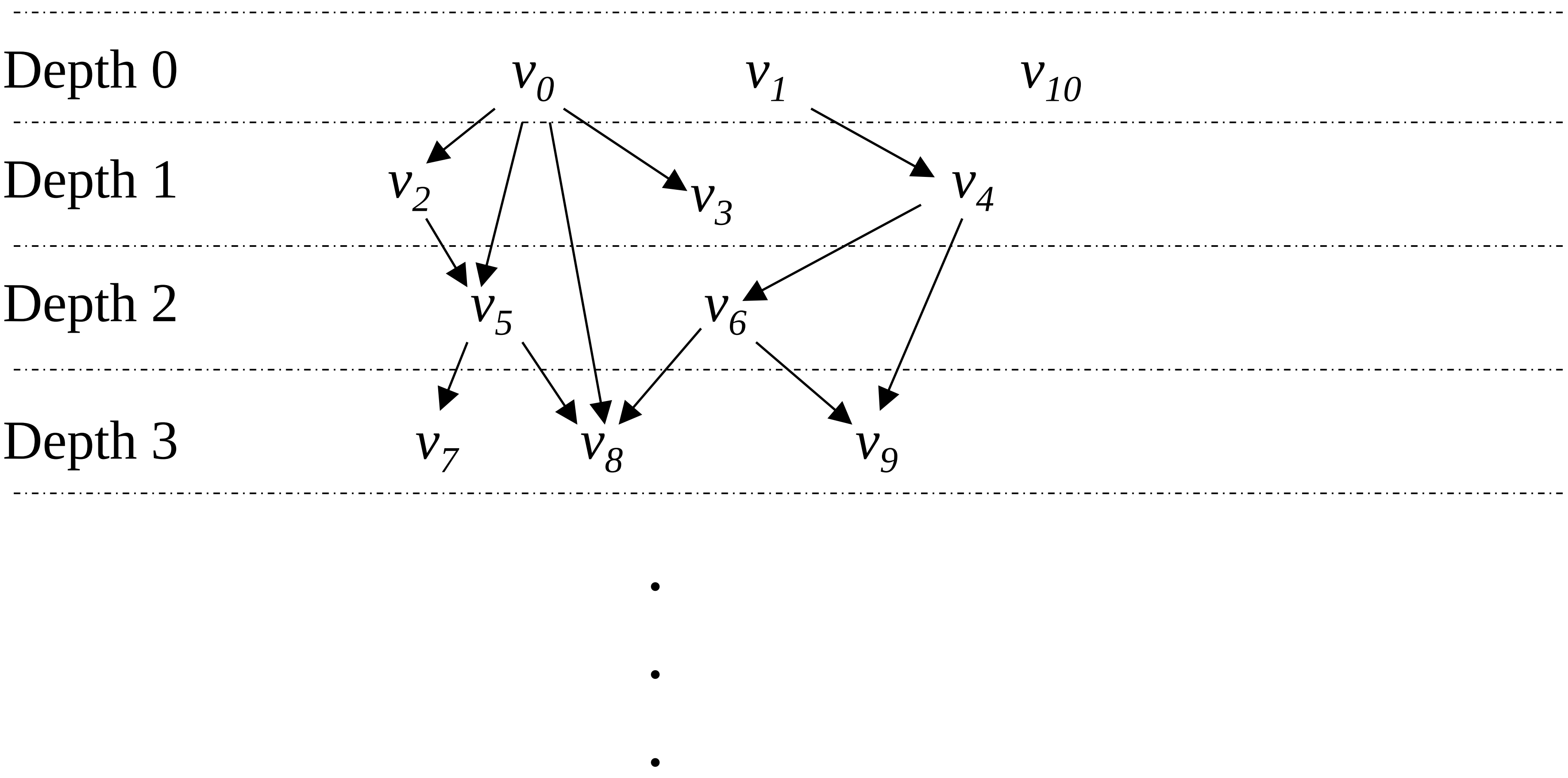
  by above, the depth is well-defined and $\leq |V| - 1$.
- *Every descendant w of a node v has higher depth than v*: If
  - $(u, \ldots, v)$ is path of length $n = \text{depth}(v)$ ending at v, and
  - $(v, \ldots, w)$ is path from v to w,

  then $(u, \ldots, v, \ldots, w)$ is a path of length $> n$ ending at w, so $\text{depth}(w) > n$.

Structure of DAGs (cont'd)

- *Every node v of positive depth has a parent of depth exactly one less*:
  - Let $(u, \ldots, v', v)$ be path of length $n = \text{depth}(v)$ ending at v.
  - Then v' is a parent of v.
  - Since $(u, \ldots, v')$ has length $n - 1$, $\text{depth}(v') \geq n - 1$.
  - Since also $\text{depth}(v') < n$ (because v is a descendant of v'), $\text{depth}(v')$ is exactly $n - 1$.
- *The nodes on any path are of increasing depth*.

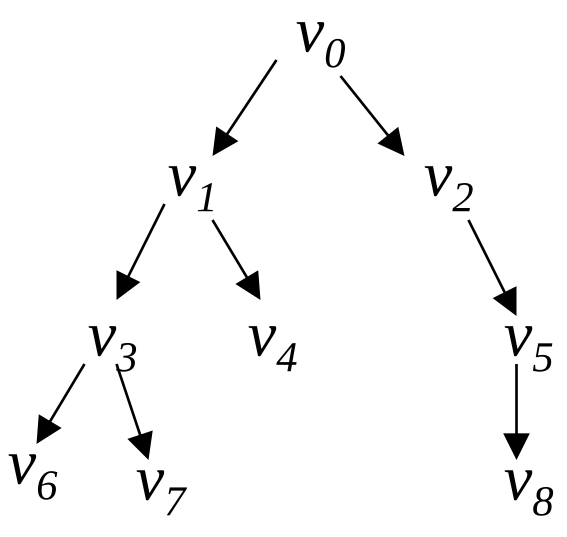
Structure of DAGs (cont'd)



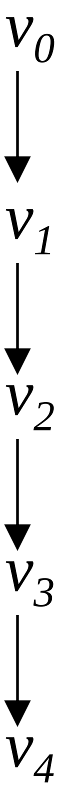
Important special cases:

- A *(rooted)* *tree* is a DAG which
  - has unique depth 0 node (the *root*), *and*
  - every other node has in-degree 1
    - (i.e. has a unique parent, of depth one less than that of the node).
- A *binary tree* is a tree in which
  - every node has out-degree at most 2.
- A *linked list* is a tree in which
  - every node has out-degree at most 1
  - or equivalently, a DAG in which ∃ at most one node of each depth

binary tree



linked list



Remarks on Depth Structure

- For *dynamic programming* algorithm
  - we need an order $v_1, v_2, \ldots, v_n$ for the vertices
    - (not a path!)

    in which parents appear before children.
  - From the above, *depth order*
    - (in which depth 0 nodes are listed first, then depth 1 nodes, etc.)

    is such an order.
  - In general there are many other such orders.
- We haven't given constructive procedure for finding the depths of all vertices.
  - For an arbitrary DAG, can be done in $O(|V| + |E|)$ time;
  - we omit algorithm, since for DAGs related to sequence analysis, the depth structure is obvious.

Weighted Directed Graphs

- A *weighted directed graph* is
  - a directed graph (V, E) together with
  - a function w from E to the real numbers,
    - i.e. with a numerical *weight* $w(e)$ (which may be positive, negative, or 0) associated to each edge e.

A weighted DAG is called a WDAG.

- The (*sum*) *weight of a path* is defined to be the sum of the weights on the edges of the path.
- Similarly, the *product weight of a path* is the product of the edge weights
  - usually only consider this when all weights are non-negative.
- weight of a path P is written $w(P)$
- For a path of length 0 (i.e. consisting of a single vertex):
  - the sum weight is 0
  - the product weight is 1

Highest Weight Paths on WDAGs

- *Problem*: find a path with the highest possible weight.
- *Solution*:
  - "Brute force" approach
    - i.e. simply enumerating all possible paths and comparing their weights

    is usually impractical (too many paths!)
  - Instead, use the method of *dynamic programming* ('The Fundamental Algorithm of Computational Biology').

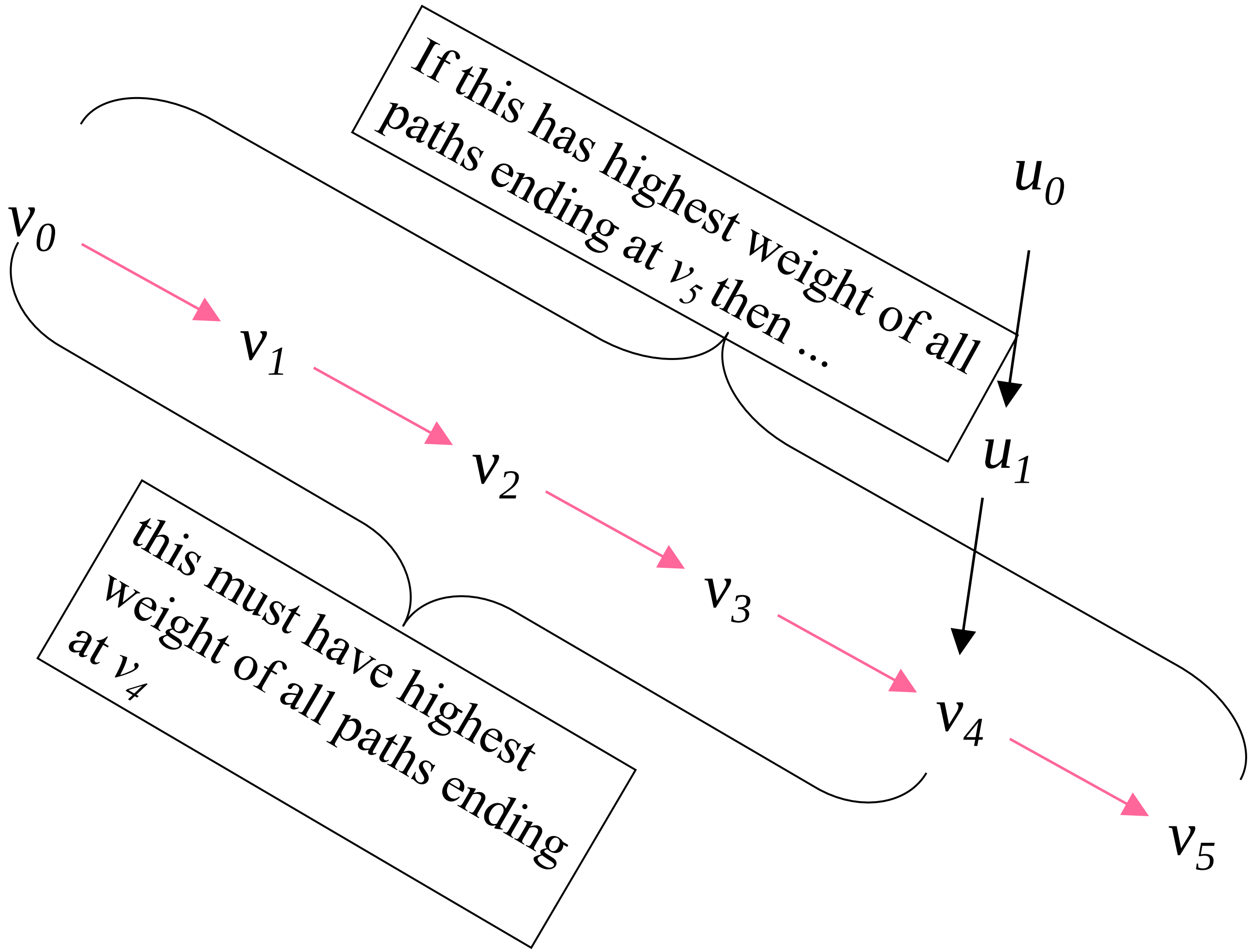
Highest Weight Paths on WDAGs (cont'd)

- Let $P_n = (v_0, v_1, \ldots, v_n)$ be a path of highest weight.
- Then for each $k < n$, the sub-path $P_k = (v_0, v_1, \ldots, v_k)$ must have highest weight of all paths ending at v_k, because
  - *if* $Q = (u_0, u_1, \ldots, v_k)$ were another path ending at v_k and having higher weight than P_k,
  - *then* the path $(Q, v_{k+1}, \ldots, v_n)$ would have weight

$$w((Q, v_{k+1}, \ldots, v_n)) = w(Q) + w((v_k, \ldots, v_n))$$
$$> w(P_k) + w((v_k, \ldots, v_n)) = w(P_n),$$

contradicting assumption that P_n has highest weight.

Subpaths of a highest-weight path can't be improved:



Highest Weight Paths on WDAGs (cont'd)

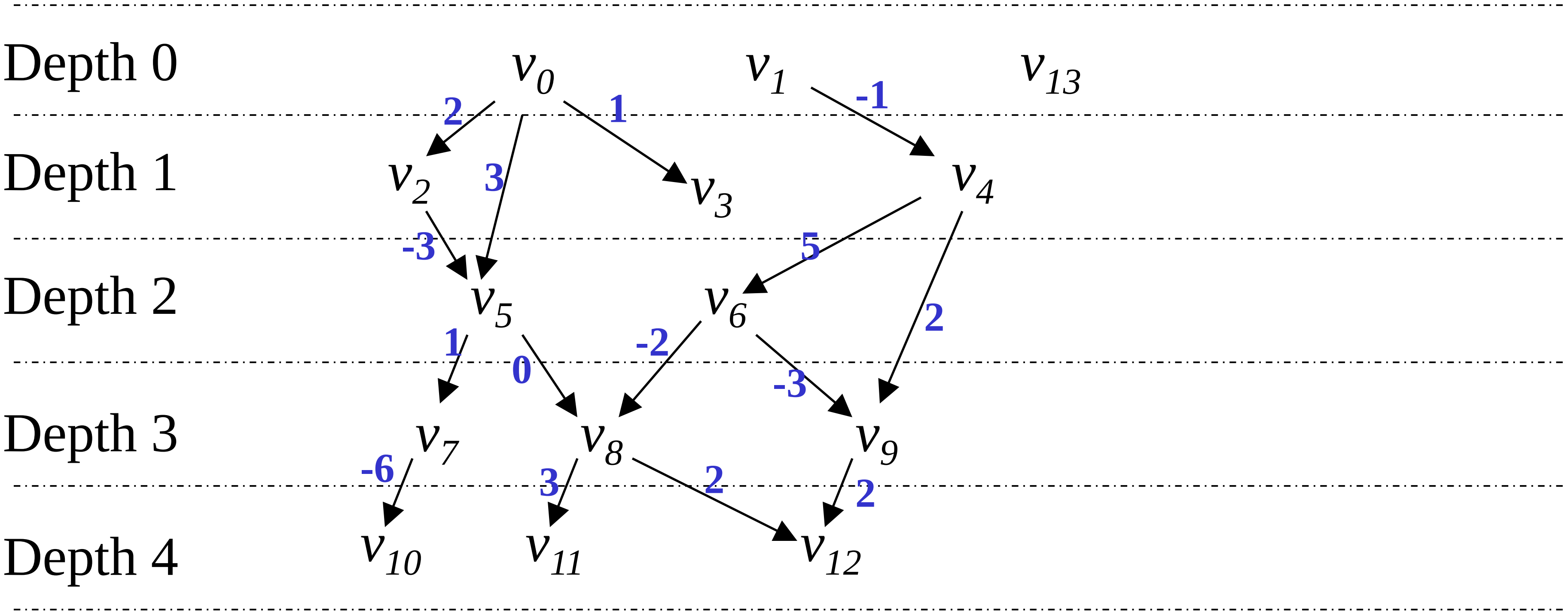
- So generalize the problem as follows:
- find, for *each* vertex v, the highest weight of all paths ending at v – call this $w(v)$
- Can find $w(v)$ in single pass through V, as follows:
  - process the v in depth order (*or any order in which parents precede children*)
  - if v has no parents, $w(v) = 0$ (the only path ending at v is (v)).
  - for any other v, except for the path (v) (which has weight 0), any path ending at v is of form $(v_0, v_1, \ldots, v_k, u, v)$. Then
  - u is a parent of v, so $w(u)$ has already been computed, and

    $$w((v_0, v_1, \ldots, v_k, u, v)) \leq w(u) + w((u,v))$$

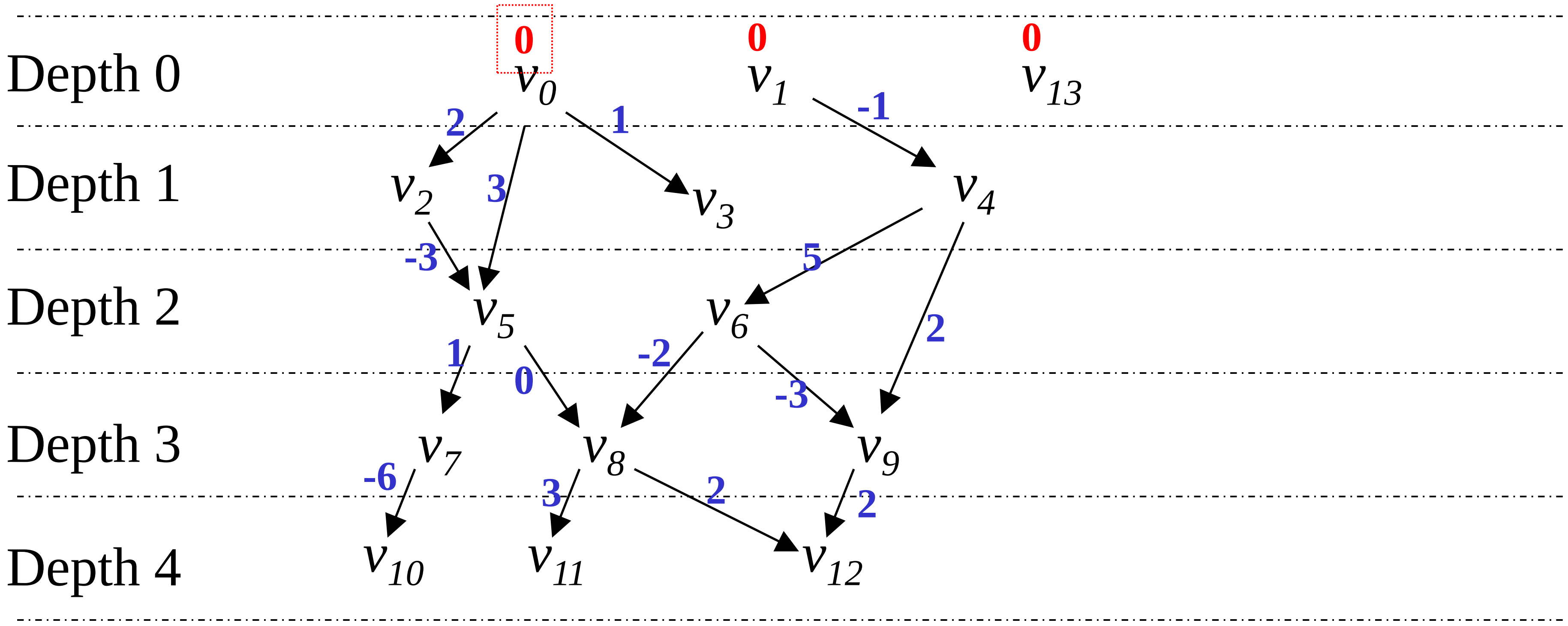
    with equality for an appropriate choice of v_i.
  - Therefore we may compute $w(v)$ as

$$w(v) = \max(0, \max_{u \in parents(v)} (w(u) + w((u,v))))$$

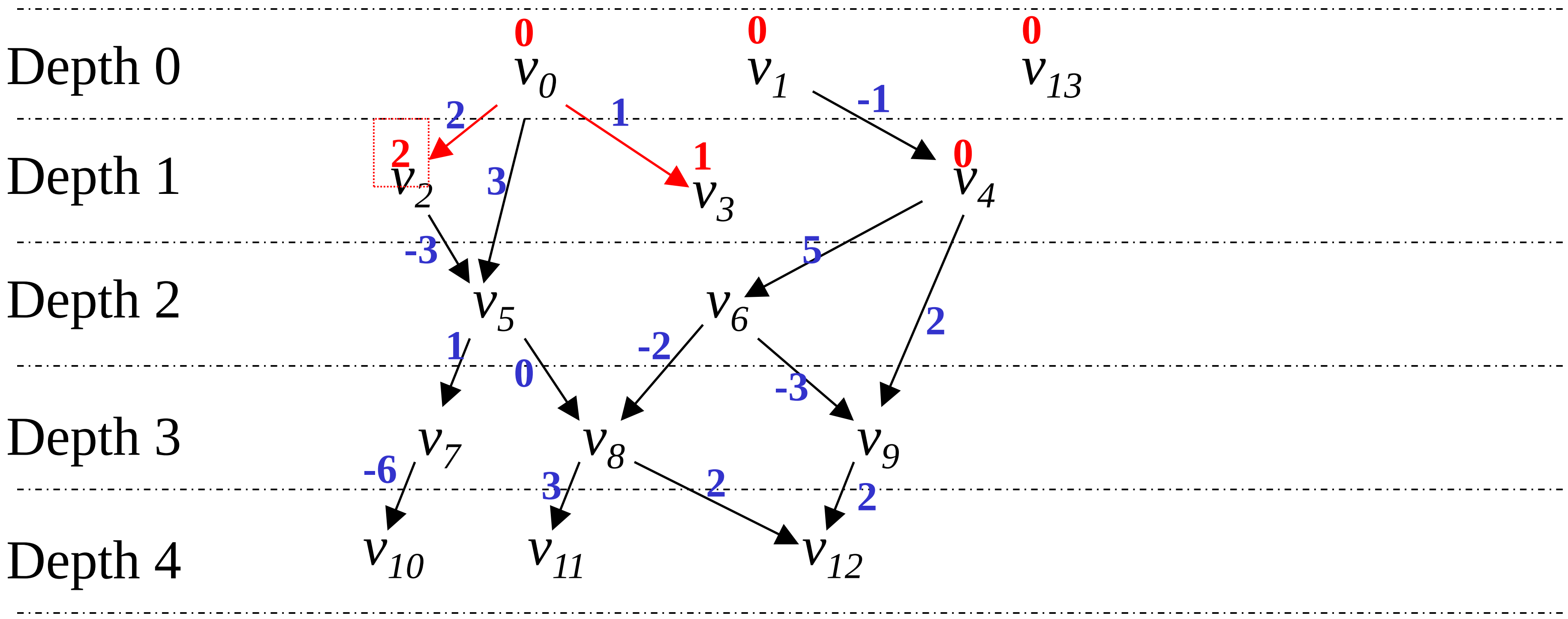
Example


Depth 0
Depth 1
Depth 2
Depth 3
Depth 4
v0
v1
v13
v2
v3
v4
v5
v6
v7
v8
v9
v10
v11
v12
2
1
3
-1
-3
5
2
1
0
-2
-3
-6
3
2
2

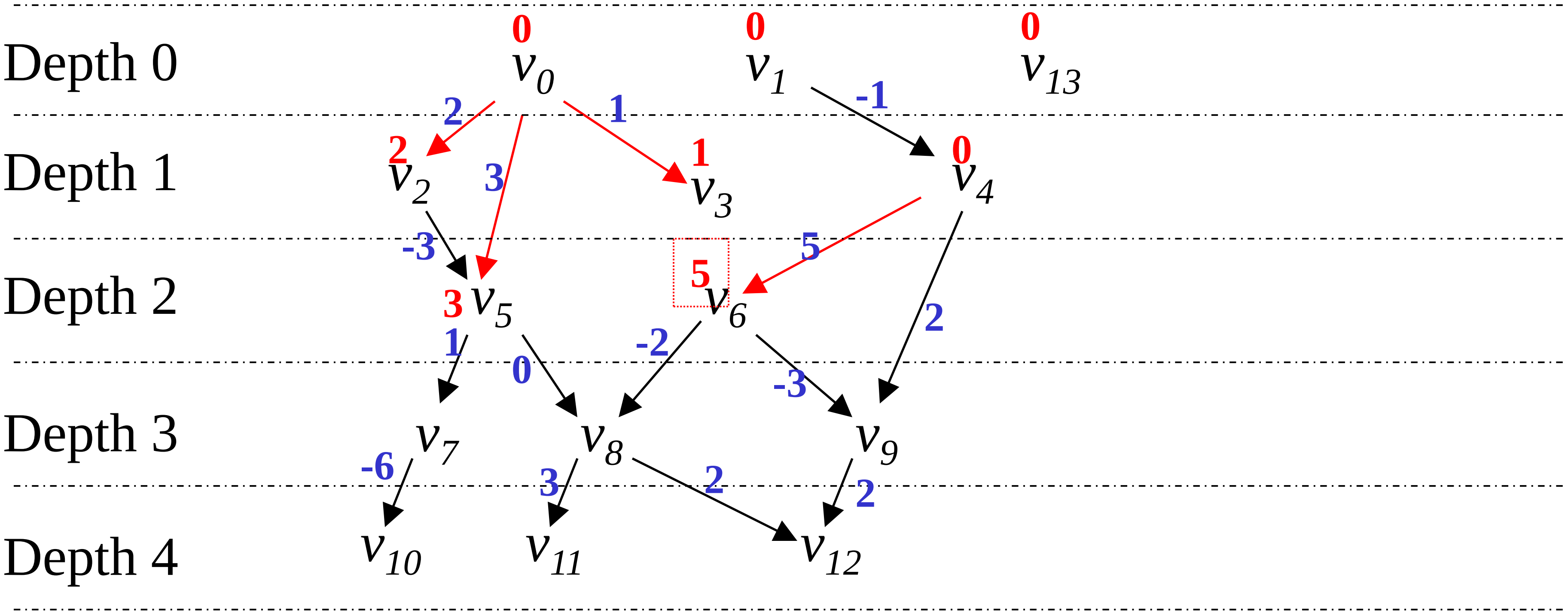
$w(v)$ – depth 0 nodes



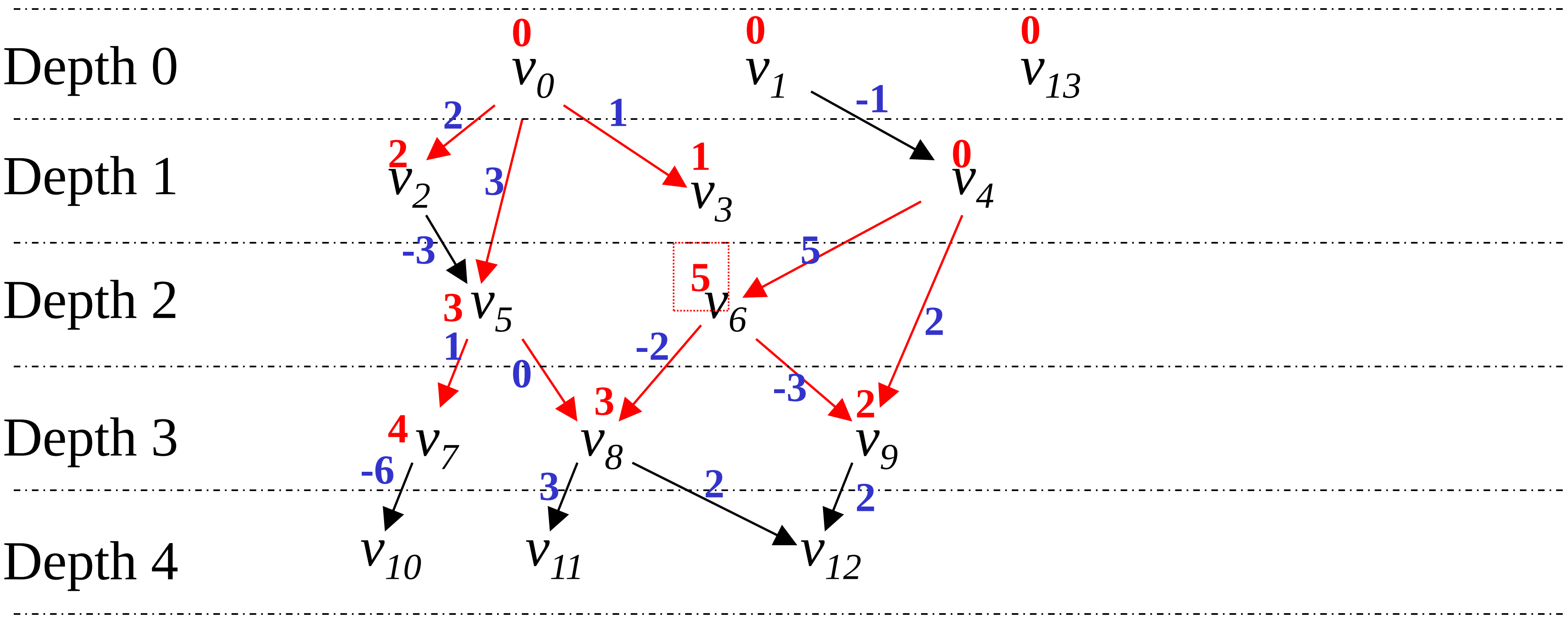
$w(v)$ – depth 1 nodes



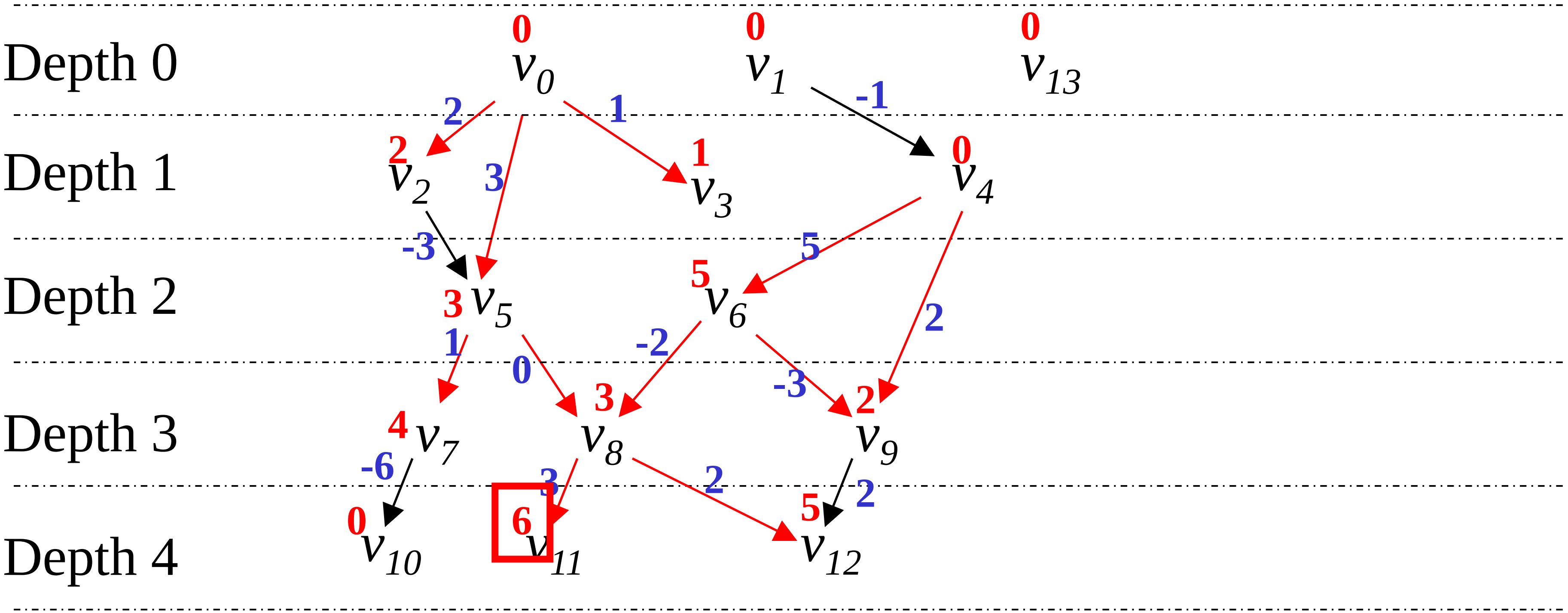
$w(v)$ – depth 2 nodes



$w(v)$ – depth 3 nodes



$w(v)$ – depth 4 nodes



Highest Weight Paths on WDAGs (cont'd)

- To reconstruct best path, need "traceback" pointer to immediate predecessor of *v* in best path:

$$T(v) = \begin{cases} v & w(v) = 0 \\ \underset{u \in \text{parents}(v)}{\arg\max} \; (w(u) \; + \; w((u,v)) & w(v) \neq 0 \end{cases}$$

  – in preceding graph, $T(v)$ is the *parent* on *red edge* coming into *v*
    - if more than one such edge, pick one at random;
    - if no such edge, $T(v) = v$

- Sometimes useful to record *beginning* of best path:

$$B(v) = \begin{cases} v & w(v) = 0 \\ B(T(v)) & w(v) \neq 0 \end{cases}$$

Highest Weight Paths on WDAGs (cont'd)

- Then highest weight of any path in graph is

$$\max_{v \in V}(w(v))$$

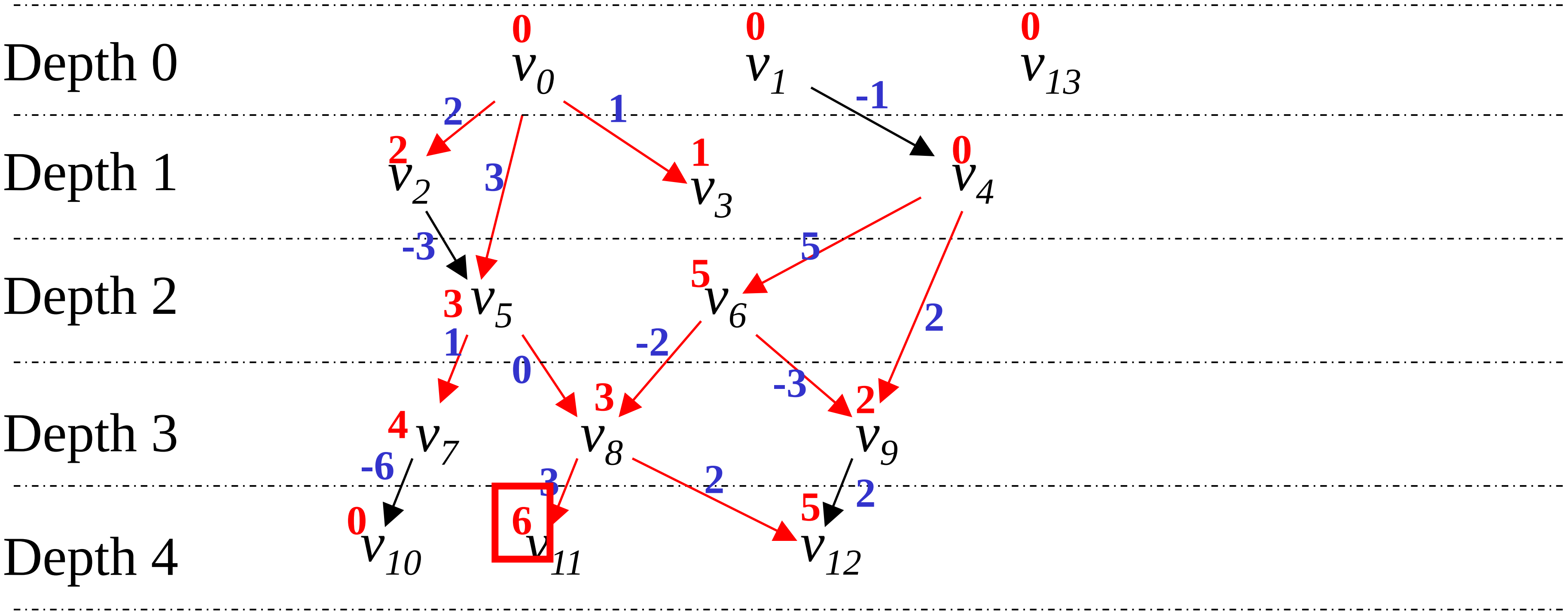
  - updated as each node is visited
    - indicated by $\boxed{}$ in preceding graph –

  and so doesn't require additional pass through vertices

- if $u = \text{argmax}_{v \in V}(w(v))$, can reconstruct highest weight path by tracing back from u, using T:
  - path ends at u;
  - immediate predecessor of u is $T(u)$;
  - predecessor of $T(u)$ is $T(T(u))$; etc.
  - stop when $T(v) = v$.

- In preceding example, highest weight is 6 and $u = v_{11}$

Dynamic programming on WDAGs



Complexity of Dynamic Programming

- Time to find a best path is $O(|E|+|V|)$:
  - in initial pass, visit each node, and each edge into that node: $O(|E|+|V|)$
  - in traceback, visit subset of nodes, and unique edge from each node: $O(|V|)$

(Complexity to find *all* highest weight paths can be higher)

For very large graphs, even $O(|E|+|V|)$ may be unacceptable!

Complexity Analysis (cont'd)

- Space requirements:
  - If only want *weight* of best path, and beginning and end, then
    - don't need $T(v)$, and
    - only need retain $w(v)$ and $B(v)$ until have processed all children of v (or when best path found so far ends at v).

    Space depends on graph structure, but usually $\ll O(|V|)$.
  - If want path itself, must store $T(v)\ \forall\ v$
    - space = $O(|V|)$
    - $\exists$ algorithms (for some graphs) to reduce this, but may take more time.